Two-point correlation properties of stochastic splitting processes

Andrea Gabrielli[1,2] and Michael Joyce[3]

[1]*Statistical Mechanics and Complexity Center, INFM-CNR, Physics Department, University "La Sapienza" of Rome,*
Piazzale Aldo Moro 2, 00185 Rome, Italy
[2]*Istituto dei Sistemi Complessi, CNR, via dei Taurini 19, 00185 Rome, Italy*
[3]*Laboratoire de Physique Nucléaire et de Hautes Energies, UMR-7585, Université Pierre et Marie Curie-Paris 6,*
75252 Paris Cedex 05, France

We study how the two-point density correlation properties of a point particle distribution are modified when each particle is divided, by a stochastic process, into an equal number of identical "daughter" particles. We consider generically that there may be nontrivial correlations in the displacement fields describing the positions of the different daughters of the same "mother" particle and then treat separately the cases in which there are, or are not, correlations also between the displacements of daughters belonging to different mothers. For both cases exact formulas are derived relating the structure factor (power spectrum) of the daughter distribution to that of the mothers. An application of these results is that they give explicit algorithms for generating, starting from regular lattice arrays, stochastic particle distributions with an arbitrarily high degree of large-scale uniformity. Such distributions are of interest, in particular, in the context of studies of self-gravitating systems in cosmology.

I. INTRODUCTION

Point processes—i.e., stochastic spatial distributions of identical point particles—provide a very useful mathematical scheme for many different N-body and granular physical systems such as crystals (both regular and perturbed) [1–3], quasicrystals [4], structural glasses, fluids [5], and self-gravitating systems in astrophysics and cosmology (see, e.g., [6–8]). They find also many applications in a wide range of scientific fields: computer image problems [9] and biometrical studies [10] are only some examples of systems which are usually represented as specific point processes with appropriate spatial correlation properties. The extension of knowledge about this class of stochastic processes can therefore be of fundamental importance for the description and analysis of many scientific topics. Indeed, this is the reason why considerable mathematical effort has already been invested in the study of this class of systems, and many useful results have been derived (e.g., see [11–13]).

In this paper we present the equations for the structure factor (or power spectrum) of a point process obtained as follows. We start with an arbitrary uniform spatial point particle distribution with a known structure factor (SF). We now suppose that each of these "mother" particles splits into a cloud of m identical "daughter" particles, where m is a cloud-independent constant. Each daughter particle is then assumed to be displaced from its mother position by a stochastic displacement which may, or may not, be correlated with the displacements of other particles. In other words each set of m particles initially lying at the same spatial point "explodes," forming a "cloud" of particles around it. For this reason we call the point process so generated a *cloud process*.

We suppose that the displacements applied to different particles belonging to the *same* mother are symmetrically distributed with arbitrary pair correlations. One can choose,

for instance, these correlations in order to fix certain moments of the mass dispersion of each cloud. We will distinguish in the following between the two cases in which the displacements applied to particles originating from *different* mothers are, or are not, correlated. We note that the results obtained for these cases are generalizations of those obtained in [14], where the case of a single daughter for each mother was analyzed and solved exactly.

The main immediate application we have in mind of the present study, and the one we discuss at some length, is the determination of the constraints on the stochastic displacement field and the number of daughters, m, required to produce a target behavior of the SF at large scale (i.e., small wave number k) in the daughter particle distribution. More specifically we are interested in the case of a large-scale SF inherent to *superhomogeneous* (or *hyperuniform*) stochastic point particle distributions [15–17]. This class of distribution is characterized by the convergence of the SF to zero as $k \rightarrow 0$. We will show explicitly that, for the case that the mother distribution is a regular lattice array, the associated (positive) exponent in the $k \rightarrow 0$ limit of the SF is related to the conservation of the local mass moments in the passage from the single point particle to the cloud of daughter particles.[1]

An interesting application of this kind of superhomogeneous point pattern consists in the preparation of initial conditions for gravitational N-body simulations [16,19,20]. In particular, it is predicted theoretically (see, e.g., [7]) that, when the SF decays at small wave numbers k faster than k^4,

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[1]We note that approximate heuristic derivations of some of these results can be found in the cosmological literature (see, e.g., [7,18]) in which constructions of this kind are considered in discussions of "causality bounds"—i.e., bounds imposed on the large-scale behavior of the SF by the existence of a causal horizon in cosmological models.

the gravitational collapse dynamics changes qualitatively compared to the case when the initial conditions are less uniform. In order to study this question through numerical simulations, it is essential to have algorithms which can generate such particle configurations, with arbitrary behavior of the SF at large wavelengths. The analysis of the particle distributions presented in this paper provides precisely such an algorithm.

The paper is structured as follows. In the next section we introduce our notation and the essential definitions, and then derive a general expression for the SF of the daughter distribution, before averaging over the realizations of the cloud process. In Sec. III we derive our general result for the SF for the case that there are only correlations between displacements of particles derived from the same mother. In Sec. IV we apply this result to the specific case that the mother distribution is a regular lattice and derive the small-k behavior of the SF of the daughter distribution. In Sec. V we then derive our result for the more general case that there is also arbitrary correlation between the displacements of daughters of different mothers and then also consider the specific case of an initial regular lattice. As an example we derive the small-k behavior of the SF of a lattice of correlated dipoles. In the final section we summarize our results and discuss briefly both possible developments and further applications of the results reported here.

II. STRUCTURE FACTOR OF A STOCHASTIC CLOUD PROCESS

We start with a spatial distribution of M mother particles in a cubic volume V, for which we write the microscopic particle density as

$$n(\mathbf{x}) = \sum_{i=1}^{M} \delta(\mathbf{x} - \mathbf{x}_i). \qquad (1)$$

Let us denote by $\langle \cdots \rangle$ the average over the ensemble of realizations of this point process, which we assume to be uniform at large scales—i.e., to have a well-defined (positive) mean particle density n_0 in the limit $V \to \infty$. The SF (or power spectrum) is then defined (see, e.g., [8]) as

$$S_n(\mathbf{k}) = \lim_{V \to \infty} \frac{\langle |\tilde{n}(\mathbf{k}; V)|^2 \rangle}{M} - (2\pi)^d n_0 \delta(\mathbf{k}), \qquad (2)$$

where the limit $V \to \infty$ is taken at fixed n_0 and

$$\tilde{n}(\mathbf{k}; V) = \int_V d^d x \, n(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} = \sum_{i=1}^{M} e^{-i\mathbf{k}\cdot\mathbf{x}_i}$$

is the Fourier transform (FT) of the point distribution in the volume V. Note that, defined in this way, $S_n(\mathbf{k} \to \infty) = 1$, which is the usual normalization of the shot noise present in any particle distribution at short distances.[2] We suppose further that the point distribution is statistically spatially homo-
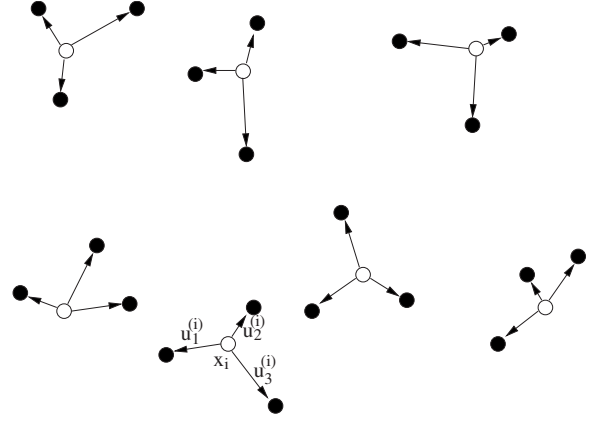

FIG. 1. Pictorial representation of the generation of a cloud process, as described in the text and characterized by Eq. (4). Starting from a mother point process (white particles), a new particle distribution is generated by the "explosion" of each mother particle into m (here, $m=3$) daughters, each of which is displaced from the original point by a stochastic displacement $\mathbf{u}$. The problem we solve is the determination of the two-point correlation properties of the new particle distribution given those of the mother distribution and the statistical properties of the displacement fields.

geneous. In this case the SF is related by a FT to the usual two-point correlation function. More specifically, if we denote by $h_n(\mathbf{x})$ the off-diagonal part of the reduced two-point correlation function, we have

$$\langle n(\mathbf{x}_0)n(\mathbf{x}_0 + \mathbf{x})\rangle - n_0^2 = n_0\delta(\mathbf{x}) + n_0^2 h_n(\mathbf{x})$$

$$= n_0 \int \frac{d^d k}{(2\pi)^d} S_n(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \qquad (3)$$

In other words,

$$S_n(\mathbf{k}) = 1 + n_0 \tilde{h}_n(\mathbf{k}),$$

with $\tilde{h}_n(\mathbf{k}) = \text{FT}[h_n(\mathbf{x})]$.

Each particle in this distribution is a mother point which splits into $m \geq 1$ particles (daughters). We take these latter to have identical unitary mass.[3] The daughter particle distribution, which we call a *cloud process*, thus clearly has an average particle density $\rho_0 = m n_0$.

Let us denote by $\mathbf{u}_r^{(i)}$ the displacement from the mother position of the rth particle ($1 \leq r \leq m$) belonging to the cloud generated by the ith mother ($1 \leq i \leq M$) of the distribution $n(\mathbf{x})$. The resulting microscopic density of particles is then (see Fig. 1)

$$\rho(\mathbf{x}) = \sum_{r=1}^{m} \sum_{i=1}^{M} \delta(\mathbf{x} - \mathbf{x}_i - \mathbf{u}_r^{(i)}). \qquad (4)$$

Let us now take the FT of $\rho(\mathbf{x})$ in the volume V:

$$\widetilde{\rho}(\mathbf{k};V) = \int_V d^d x \, \rho(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} = \sum_{r=1}^{m} \sum_{i=1}^{M} e^{i\mathbf{k}\cdot(\mathbf{x}_i+\mathbf{u}_r^{(i)})}.$$

We can now write the SF for $\rho(\mathbf{x})$ as

$$S_\rho(\mathbf{k}) = \lim_{V\to\infty} \frac{\langle\overline{|\widetilde{\rho}(\mathbf{k};V)|^2}\rangle}{N} - (2\pi)^d \rho_0 \delta(\mathbf{k}), \qquad (5)$$

where $\langle\cdots\rangle$ is, as above, the average over the realizations of $n(\mathbf{x})$ and $\overline{(\cdots)}$ is the average over the realizations of the displacement field. We will always assume here that the displacement field is statistically independent of the realization of the mother distribution, so that these two averages commute. In general, we can write

$$|\widetilde{\rho}(\mathbf{k};V)|^2 = N + \sum_{i=1}^{M} \sum_{r\neq s}^{1,m} e^{-i\mathbf{k}\cdot(\mathbf{u}_r^{(i)}-\mathbf{u}_s^{(i)})} + \sum_{i\neq j}^{1,M} \sum_{r,s}^{1,m} e^{-i\mathbf{k}\cdot(\mathbf{x}_i+\mathbf{u}_r^{(i)}-\mathbf{x}_j-\mathbf{u}_s^{(j)})},$$
$$(6)$$

where $\Sigma_{r\neq s}$ denotes the sum over r and s, excluding the diagonal terms $r=s$, and similarly for $\Sigma_{i\neq j}$.

III. UNCORRELATED CLOUDS: GENERAL RESULT

In this section we assume the following statistical properties for the displacement field: (i) the displacements applied to daughters with different mothers are statistically independent and (ii) the displacements applied to different daughters with the same mother may be arbitrarily correlated.

More precisely, let us denote by $p(\mathbf{u})$ the probability density function (PDF) for a single displacement, by $p_s(\mathbf{u},\mathbf{v})$ the joint PDF of the displacements $\mathbf{u}$ and $\mathbf{v}$ applied to two daughters of the same mother, and by $p_d(\mathbf{u},\mathbf{v};\mathbf{x})$ the joint PDF of the displacements applied to two particles belonging to different mothers separated by $\mathbf{x}$. Our assumptions imply that we have $p_d(\mathbf{u},\mathbf{v};\mathbf{x})=p(\mathbf{u})p(\mathbf{v})$ for any $\mathbf{x}\neq\mathbf{0}$, while we allow $p_s(\mathbf{u},\mathbf{v})\neq p(\mathbf{u})p(\mathbf{v})$.

Note that in writing the PDFs in this way, without labels for the clouds and for the particles in a single cloud to which the displacements apply, we assume implicitly the following symmetries: (i) $p(\mathbf{u})$ is the same for all displacements and (ii) $p_s(\mathbf{u},\mathbf{v})$ does not depend on the cloud and is the same for all $m(m-1)/2$ couples of m particles belonging to the same cloud. In other words, if $P(\mathbf{u}_1,\ldots,\mathbf{u}_m)$ is the joint PDF of the displacements applied to the m daughter particles in a cloud, we assume that $P(\mathbf{u}_1,\ldots,\mathbf{u}_m)$ is the same for all clouds and is invariant under any permutation of the m displacement variables. This implies in particular that $p_s(\mathbf{u},\mathbf{v})=p_s(\mathbf{v},\mathbf{u})$ and, as we show below, important constraints on the displacement-displacement correlations inside a single cloud. We clearly also have the consistency condition

$$\int d^d v \, p_s(\mathbf{u},\mathbf{v}) = p(\mathbf{u}).$$

With these assumptions it is simple to show that it follows from Eq. (6) that

$$\overline{|\widetilde{\rho}(\mathbf{k})|^2} = N + N(m-1)\widetilde{p}_s(\mathbf{k},-\mathbf{k}) + m^2|\widetilde{p}(\mathbf{k})|^2 \sum_{i\neq j}^{1,M} e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)},$$
$$(7)$$

where $\widetilde{p}(\mathbf{k})=\mathrm{FT}[p(\mathbf{u})]$ (i.e., the characteristic function of the single stochastic displacement) and $\widetilde{p}_s(\mathbf{k},-\mathbf{k})$ is the following diagonal two-point FT of $p_s(\mathbf{u},\mathbf{v})$:

$$\widetilde{p}_s(\mathbf{k},-\mathbf{k}) = \int d^d u \, d^d v \, p_s(\mathbf{u},\mathbf{v}) e^{-i\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})}.$$

Note that

$$\widetilde{p}_s(\mathbf{k},-\mathbf{k}) = \widetilde{\phi}_s(\mathbf{k}) = \int d^d w \, \phi_s(\mathbf{w}) e^{-i\mathbf{k}\cdot\mathbf{w}},$$

where $\phi_s(\mathbf{w})$ is the PDF of the nth relative displacement $\mathbf{w} = (\mathbf{u}-\mathbf{v})$ between two particles in the same cloud:

$$\phi_s(\mathbf{w}) = \int d^d u \, d^d v \, p_s(\mathbf{u},\mathbf{v}) \delta(\mathbf{w}-\mathbf{u}+\mathbf{v}).$$

Since $p_s(\mathbf{u},\mathbf{v})=p_s(\mathbf{v},\mathbf{u})$, $\phi_s(\mathbf{w})$ is an even function of $\mathbf{w}$, and, as a consequence, $\widetilde{\phi}_s(\mathbf{k})$ is a real function. Further, from the fact that $\phi_s(\mathbf{w})$ is a PDF, it follows that $\widetilde{\phi}_s(\mathbf{k}=\mathbf{0})=1$ and $|\widetilde{\phi}_s(\mathbf{k})|\leq 1$ at all $\mathbf{k}$.

To perform the average $\langle\cdots\rangle$ (over the realizations of mother distribution), we use that

$$\sum_{i\neq j}^{1,M} e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)} = \sum_{i,j}^{1,M} e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)} - M$$

and that from Eq. (2), for asymptotically large M (i.e., for $V\to\infty$ with n_0 fixed), one has

$$\left\langle \sum_{i,j}^{1,M} e^{-i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)} \right\rangle = M[S_n(\mathbf{k}) + (2\pi)^d n_0 \delta(\mathbf{k})].$$

Using these results, together with $N=mM$ and $\rho_0=mn_0$, we finally obtain

$$S_\rho(\mathbf{k}) = 1 + (m-1)\widetilde{\phi}_s(\mathbf{k}) - m|\widetilde{p}(\mathbf{k})|^2 + m|\widetilde{p}(\mathbf{k})|^2 S_n(\mathbf{k}).$$
$$(8)$$

This is the principal result of this section. It is useful to write it also in the form

$$S_\rho(\mathbf{k}) = S_\rho^{(0)}(\mathbf{k}) + m|\widetilde{p}(\mathbf{k})|^2 S_n(\mathbf{k}), \qquad (9)$$

where

$$S_\rho^{(0)}(\mathbf{k}) = 1 + (m-1)\widetilde{\phi}_s(\mathbf{k}) - m|\widetilde{p}(\mathbf{k})|^2 \qquad (10)$$

depends only on the statistical properties of the displacement fields (i.e., independent of those of the mother distribution).

We note that for the case $m=1$, Eq. (8) reduces to

$$S_1(\mathbf{k}) = 1 - |\widetilde{p}(\mathbf{k})|^2 + |\widetilde{p}(\mathbf{k})|^2 S_0(\mathbf{k}). \qquad (11)$$

This is precisely the equation derived in [8,14] for the transformation of the PS of a point particle distribution from $S_0(\mathbf{k})$ to $S_1(\mathbf{k})$ when each particle is randomly displaced,

independently of the others, with a PDF $p(\mathbf{u})$ for the single random displacements.

Another simple case is that in which the mother particle distribution $n(\mathbf{x})$ is itself completely uncorrelated with $h_n(\mathbf{x}) = 0$—i.e., generated by a homogeneous Poisson process. Then $S_n(\mathbf{k}) = 1$, and consequently Eq. (8) gives

$$S_\rho(\mathbf{k}) = 1 + (m-1)\tilde{\phi}_s(\mathbf{k}).$$

Since $\tilde{\phi}_s(\mathbf{k} \to 0) = 1$, it follows that $S_\rho(\mathbf{k} \to 0) = m$; i.e., at large scales the cloud process is identical to the original distribution (up to a change in the mean particle density by the factor m). This simply translates the fact that if a point process has no correlation, we cannot create correlation at large scales by dividing the particles into clouds by a stochastic process which incorporates no correlation between the different clouds.

A. Properties of $S_\rho(\mathbf{k})$

By definition, any SF, and therefore the one we have derived, must satisfy the conditions

$$S_\rho(\mathbf{k}) \geq 0, \tag{12}$$

$$S_\rho(\mathbf{k} \to \infty) = 1. \tag{13}$$

These must hold for *any* input $S_n(\mathbf{k})$ (itself obeying these properties) and any finite value of m.

The second property, Eq. (13), is simple to verify. It follows from the fact that both $\tilde{\phi}_s(\mathbf{k})$ and $|\tilde{p}(\mathbf{k})|$ vanish in the large-k limit. This is the case because they are FTs of functions which are integrable at the origin.

That the first property, Eq. (12), must be satisfied by our result is trivial: we obtained $S_\rho(\mathbf{k})$ by simply averaging Eq. (6) which is non-negative definite by construction. However, as we now discuss, it is not as simple as one might anticipate to verify it directly from Eq. (8). The property (12) in fact encodes in a concise and very nontrivial manner constraints on the joint PDF $p_s(\mathbf{u}, \mathbf{v})$ which follow from the assumption that it is unique for any two particles in a cloud.

First we note that Eq. (12) holds in fact if and only if

$$S_\rho^{(0)}(\mathbf{k}) \geq 0 \tag{14}$$

for all $\mathbf{k}$. This is the case because Eq. (12) must be true for an arbitrary $S_n(\mathbf{k})$ and it is always possible to choose a mother point process for which it vanishes at any given $\mathbf{k}$ (taking, e.g., an appropriate regular lattice). For $m = 1$ this condition is trivially satisfied, as $|\tilde{p}(\mathbf{k})| \leq 1$ by definition (as FT of a PDF). For $m \geq 2$ it may be rewritten as the condition

$$\tilde{\phi}_s(\mathbf{k}) \geq -\frac{1}{m-1} + \frac{m}{m-1}|\tilde{p}(\mathbf{k})|^2 > -1. \tag{15}$$

This does not trivially follow from the fact that $\tilde{p}(\mathbf{k})$ and $\tilde{\phi}_s(\mathbf{k})$ are FTs of PDFs. As noted above, the latter give only the weaker condition $\phi_s(\mathbf{k}) \geq -1$. Clearly Eq. (15) encodes a nontrivial constraint on $\phi_s(\mathbf{k})$, arising from the fact that it is related to a PDF for the *joint* displacement PDF. The latter is not simply an arbitrary normalizable function. The condition

(15) tells us that we have in fact constrained it mathematically by the assumption about it we have made in our derivation: we have assumed that the cloud is generated in such a way that the joint two-displacement PDF is identical for all couples of particles.

To illustrate this more explicitly we derive now the form taken by the constraint, in the form of Eq. (14), when $S_\rho^{(0)}(\mathbf{k})$ is expanded in Taylor series around $\mathbf{k} = 0$. Such an expansion can be made assuming that both $p(\mathbf{u})$ and $\phi_s(\mathbf{w})$ are rapidly decreasing at large u and w so that their FTs are analytic at $\mathbf{k} = 0$. We then have

$$\tilde{p}(\mathbf{k}) = \sum_{l=0}^{\infty} (-i)^l \frac{\overline{(\mathbf{k} \cdot \mathbf{u})^l}}{l!} \tag{16}$$

and

$$\tilde{\phi}_s(\mathbf{k}) \equiv \tilde{p}_s(\mathbf{k}, -\mathbf{k}) = \sum_{l=0}^{\infty} (-i)^l \frac{\overline{(\mathbf{k} \cdot \mathbf{w})^l}}{(l)!}. \tag{17}$$

The condition (14) can therefore be rewritten as

$$S_\rho^{(0)}(\mathbf{k}) = \sum_{l=1}^{\infty} \frac{(-i)^l}{l!} \left\{ (m-1)\overline{[\mathbf{k} \cdot (\mathbf{u} - \mathbf{v})]^l} \right.$$
$$\left. - m \sum_{l'=0}^{l} \binom{l}{l'} \overline{(\mathbf{k} \cdot \mathbf{u})^{l-l'}} \times \overline{(\mathbf{k} \cdot \mathbf{v})^{l'}} \right\} \geq 0, \tag{18}$$

where we have written explicitly $\mathbf{w} = (\mathbf{u} - \mathbf{v})$ and used the symmetry assumption $\overline{(\mathbf{k} \cdot \mathbf{v})^j} = \overline{(\mathbf{k} \cdot \mathbf{u})^j}$ for any j. This inequality, valid for all $\mathbf{k}$, fixes all the constraints on the two-displacement correlation function in any cloud.

Using again the fact that $\phi_s(\mathbf{w}) = \phi_s(-\mathbf{w})$ and making the further assumption that $p(\mathbf{u}) = p(-\mathbf{u})$, all the odd-power terms in this expression vanish, so that we obtain

$$S_\rho^{(0)}(\mathbf{k}) = \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l)!} \left\{ (m-1)\overline{[\mathbf{k} \cdot (\mathbf{u} - \mathbf{v})]^{2l}} \right.$$
$$\left. - m \sum_{l'=0}^{l} \binom{2l}{2l'} \overline{(\mathbf{k} \cdot \mathbf{u})^{2(l-l')}} \times \overline{(\mathbf{k} \cdot \mathbf{v})^{2l'}} \right\} \geq 0. \tag{19}$$

The leading term dominates at sufficiently small k and therefore has to be non-negative. This implies the following constraint on the correlations of any two displacements in the set of m correlated random displacements in each cloud: the matrix

$$c_{\mu\nu} \equiv (m-1)\overline{u^{(\mu)}v^{(\nu)}} + \overline{u^{(\mu)}u^{(\nu)}}, \tag{20}$$

where $\mu, \nu = 1, \ldots, d$, has to be non-negative definite. In our analysis in the next section, of the case that the mother distribution is a lattice, we will see how this constraint, and ones which can derived at subsequent order in this expansion, simplifies and is explicitly verified in certain cases.

B. Behavior at small k

Let us now consider specifically the properties of the SF of the mother and daughter distributions as $\mathbf{k} \to 0$.

We note first that, because of the normalization conditions on the PDFs of the displacements, both $\widetilde{\phi}_s(\mathbf{k})$ and $|\widetilde{p}(\mathbf{k})|$ converge continuously to unity as $\mathbf{k} \to \mathbf{0}$. It follows from Eq. (10) that $S_\rho^{(0)}(\mathbf{k} \to \mathbf{0}) \sim k^\alpha$, where $\alpha > 0$. Supposing now that the initial (mother) point distribution has $S_n(k \to 0) \sim k^\gamma$, we can infer that $S_\rho(k \to 0) \sim k^{\gamma'}$ where (i) $\gamma' = \gamma$ for $\gamma \le 0$ and (ii) $\gamma' = \min\{\gamma, \alpha\}$ for $\gamma > 0$.

Thus the exponent of the SF around $k = 0$ *can never be larger* in the cloud process than in the original mother point process. Further, it may differ from it (in which case it is smaller) only if $S_n(k=0) = 0$. Note that these conclusions hold independently of any assumption about the cloud process, other than that there is no correlation between the displacement sets creating different clouds and that the displacements are symmetrically distributed as shown above.

This result may be explained more physically as follows. The exponent of the small-k behavior of the SF can be considered as a measure of the degree of order in the stochastic point process at asymptotically large scales [15,17]. The greater the exponent, the more ordered is the distribution. Indeed, for any lattice, which is the class of the most ordered particle distributions, the SF vanishes identically around $\mathbf{k} = \mathbf{0}$; i.e., we can consider it to correspond to the behavior k^∞. Clearly a cloud process, without any correlation between the arrangement of matter in the different clouds, cannot increase the degree of order. That it may, on the other hand, decrease the degree of order when the mother distribution has the property that $S_n(k=0) = 0$ reflects the difference between this class of distributions and those with $S_n(k=0) > 0$. Indeed, this difference is that underlined by the classification of the former point processes as *superhomogeneous* (or *hyperuniform*): the rapid decay of the density fluctuations at long wavelengths which characterizes them are the result of a delicate balance between small-scale and large-scale correlations in direct space. Indeed, the condition $S_n(k=0) = 0$ is explicitly an integral constraint on the two-point correlation function over all space. The processes which we are considering, in which each particle "explodes" independently of all others, can break these global constraints by modifying only the small-scale correlation properties. Instead, for distributions with $S_n(k=0) \neq 0$,[4] which do not present such a correlation balance, such an uncorrelated redistribution of matter at small scales cannot modify the nature of the system at large scales.

Let us now consider the problem of the construction of a point process with a target behavior of $S(\mathbf{k} \to 0)$. From the results we have just derived it follows that a cloud process of the type we have just analyzed, without correlations between the displacements of the members of different clouds, may be used to generate a distribution with the target exponent $\alpha > 0$ provided we start with $\gamma > \alpha$. In practice, we can start with $\gamma = \infty$ by taking a regular lattice, for which $S_n(\mathbf{k}) = 0$ identically in a finite region around $k = 0$ (specifically, in the first Brillouin zone). The generated distribution will then have the exponent α, which depends through Eq. (10) only on the statistical properties of the displacement fields. Since

$\alpha > 0$, the generated process is necessarily superhomogeneous. The question we now address is what values of α are attainable and for what conditions on the number of daughters m and the displacement fields they are realized. We note that in [14] it has been shown that, for $m = 1$, one can realize by an appropriate choice of $p(\mathbf{u})$ any exponent in the range $0 < \alpha \le 2$. The upper bound $\alpha = 2$ results for any $p(\mathbf{u})$ with a finite variance, while the lower exponents are realized for PDFs with appropriately divergent moments of displacements at order less than 2. For the case of correlated displacements, with Gaussian statistics, it has been shown also in [14] (see also [21]) that a maximal value of $\alpha = 4$ may be attained. We will see now that, with an appropriate value of m and conditions on the displacement fields, arbitrarily large positive target values of α are attainable. We will show that to obtain a certain value of α requires that one fix a sufficiently large number of mass moments of all clouds of particles with respect to their respective initial position on a regular lattice.

C. Explicit expansion around $k = 0$

For this study of the small-k behavior of $S_\rho(\mathbf{k})$ we consider the expansion of both $\widetilde{\phi}_s(\mathbf{k})$ and $\widetilde{p}(\mathbf{k})$ in power series of $\mathbf{k}$, which are given, respectively, by Eqs. (17) and (16). Using these expressions in Eq. (8) we obtain

$$S_\rho(\mathbf{k}) = 1 + (m-1)\sum_{l=0}^{\infty} (-i)^l \frac{\overline{[\mathbf{k} \cdot (\mathbf{u} - \mathbf{v})]^l}}{l!}$$
$$+ m \left| \sum_{l=0}^{\infty} (-i)^l \frac{\overline{(\mathbf{k} \cdot \mathbf{u})^l}}{l!} \right|^2 [S_n(\mathbf{k}) - 1]. \quad (21)$$

All these expressions are valid only under the assumption that all the moments in the sums are finite, while the derivation of Eqs. (8) and (10) required only the integrability of the probability distributions. If the probability distributions $p(\mathbf{u})$ and $\phi_s(\mathbf{w})$ have only a finite number of finite moments, the corresponding sums must be terminated at the appropriate order. There is then an additional term, of which the leading singular part can easily be determined. We will not discuss here the case in which there are such singularities.[5] The required generalization of the analysis described here is straightforward following the procedure defined in [14].

IV. UNCORRELATED CLOUD LATTICE

Given the above motivation we now analyze in detail the cloud processes in the previous section for the particular case that the mother particle distribution is a regular lattice—i.e.,

$$n(\mathbf{x}) = \sum_{\mathbf{R}} \delta(\mathbf{x} - \mathbf{R}),$$

where $\mathbf{R}$ is the generic lattice site. In this case the SF of $n(\mathbf{x})$ is

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[4] These can be classified into Poisson-like for $S_n(k=0) < +\infty$ and long-range correlated for $S_n(k \to 0) \to +\infty$.

[5] We will see below that to obtain exponents greater than 4 with uncorrelated clouds the PDFs for the displacements must, in fact, have compact support.

$$S_n(\mathbf{k}) = (2\pi)^d n_0 \sum_{\mathbf{H} \neq 0} \delta(\mathbf{k} - \mathbf{H}), \qquad (22)$$

where the sum is over all the vectors $\mathbf{H}$ of the reciprocal lattice, but $\mathbf{H} = 0$. Note that this vanishes identically in the first Brillouin zone, and therefore in this region of $\mathbf{k}$ space the following relation holds exactly:

$$S_\rho(\mathbf{k}) = 1 + (m-1)\sum_{l=0}^{\infty} (-i)^l \frac{\overline{(\mathbf{k} \cdot \mathbf{w})^l}}{l!} - m \left| \sum_{l=0}^{\infty} (-i)^l \frac{\overline{(\mathbf{k} \cdot \mathbf{u})^l}}{l!} \right|^2.$$
$$(23)$$

To simplify the presentation of our determination of the conditions required to have an arbitrary (analytic) small-k behavior of $S_\rho(\mathbf{k})$, we start with the one-dimensional case, for which Eq. (23) becomes

$$S_\rho(k) = 1 + (m-1)\sum_{l=0}^{\infty} (-ik)^l \frac{\overline{(u-v)^l}}{l!} - m \left| \sum_{l=0}^{\infty} (-ik)^l \frac{\overline{u^l}}{l!} \right|^2,$$
$$(24)$$

where u and v are the displacements applied to two different particles belonging to the same cloud. First of all we see immediately, as above underlined, that the final particle distribution $\rho(x)$ is superhomogeneous, as the zeroth-order contribution in k to $S_\rho(k)$ vanishes identically for any choice of $p_s(u,v)$ [i.e., of $\phi_s(w)$]. In the notation of the previous section, we have explicitly that $\alpha \geq 1$. We can write

$$\left| \sum_{l=0}^{\infty} (-ik)^l \frac{\overline{u^l}}{l!} \right|^2 = \sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \overline{u^j} \times \overline{v^{l-j}},$$

where we have used u and v instead of only u as the moments of the single displacement are the same for every particle. Moreover, by expanding the terms $(u-v)^l$ in Eq. (24), we have

$$\sum_{l=0}^{\infty} (-ik)^l \frac{\overline{(u-v)^l}}{l!} = \sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \overline{u^j} \times \overline{v^{l-j}}.$$

Therefore Eq. (24) becomes

$$S_\rho(k) = \sum_{l=1}^{\infty} \frac{(-ik)^l}{l!} \sum_{j=0}^{l} (-1)^j \binom{l}{j} [(m-1)\overline{u^j} \times \overline{v^{l-j}}$$
$$- m\overline{u^j} \times \overline{v^{l-j}}]. \qquad (25)$$

Making the additional assumption of statistical symmetry in the displacements, $p(u) = p(-u)$, all the terms with odd l in Eq. (25) vanish.

A. Order-by-order analysis and conservation of mass moments ($d = 1$)

Let us now analyze in detail Eq. (25), denoting by $\mathcal{O}_n(k)$ its term proportional to k^n. Given our hypotheses the lowest-order nonzero term is $n = 2$:

$$\mathcal{O}_2(k) = [\overline{u^2} + (m-1)\overline{u \times v}]k^2.$$

It is simple to verify explicitly that $[\overline{u^2} + (m-1)\overline{u \times v}] \geq 0$ always, as required [from the fact that $S_\rho(k)$ is a SF]. First of

all it is the one-dimensional version of the condition (20). This can be seen more directly as follows: if we denote by u_i with $i = 1, \ldots, m$ the displacements applied respectively to the m daughter particles originating from the same mother (i.e., belonging to the same cloud), it is clear that

$$\overline{\left(\sum_{i=1}^{m} u_i \right)^2} \geq 0.$$

This quantity, however, given our symmetry hypotheses about the displacments distribution, is nothing other than

$$\overline{\left(\sum_{i=1}^{m} u_i \right)^2} = m[\overline{u^2} + (m-1)\overline{u \times v}].$$

Consequently the condition to have an identically vanishing $\mathcal{O}_2(k)$ term, and therefore a small-k SF of order greater than 2 (i.e., $\alpha > 2$), is $\overline{(\sum_{i=1}^{m} u_i)^2} = 0$, or in other words,

$$\sum_{i=1}^{m} u_i = 0, \qquad (26)$$

with probability 1.[6] This means that the center of mass of each cloud does not move away from the mother particle when the displacements are applied. Clearly for $m = 1$ this condition can only be trivially satisfied by applying no displacement, in which case the daughter distribution is the original lattice distribution. For $m = 2$ it can be satisfied nontrivially: choosing the displacement of a first point with the PDF $p(u)$, the other particle is then displaced deterministically by $-u$. For $m > 2$ the condition can be satisfied while admitting a higher degree of stochasticity: it fixes deterministically the displacement of only one particle among m once the other $(m-1)$ are chosen stochastically.

The analysis of the term $\mathcal{O}_4(k)$ is more complex. We will now show that the condition for it to vanish is one on the second moment of the mass dispersion of each cloud. Directly from Eq. (25) we have

$$\mathcal{O}_4(k) = -\frac{k^4}{12}[\overline{u^4} + 4(m-1)\overline{u^3 v} - 3(m-1)\overline{u^2 v^2} + 3m\sigma^4],$$
$$(27)$$

where we have denoted $\sigma^2 = \overline{u^2}$ and again used the assumed symmetries of the displacement field. If the term of $\mathcal{O}_2(k)$ vanishes at all $\mathbf{k}$—i.e., if Eq. (26) is satisfied for each cloud—then the $\mathcal{O}_4(k)$ term must be non-negative [since the SF $S_\rho(k)$ must be non-negative at all k]. Thus the coefficient of k^4 in Eq. (27) must be non-negative. In order to show that this is indeed the case we note first that Eq. (26) implies that

$$\overline{u_1^3 \sum_{i=1}^{m} u_i} = \overline{u^4} + (m-1)\overline{u^3 v} = 0.$$

By using this relation in the coefficient of Eq. (27) we then have

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[6]Note that if we had allowed an asymmetric $p(u)$ with a nonzero average value $\overline{u}$ the same condition $\mathcal{O}_2(k) = 0$ would have been written as $\sum_{i=1}^{m} u_i = m\overline{u}$ for all clouds.

$$-\overline{u^4} - 4(m-1)\overline{u^3 v} + 3(m-1)\overline{u^2 v^2} - 3m\sigma^4 = 3\overline{u^4} + 3(m$$
$$-1)\overline{u^2 v^2} - 3m\sigma^4.$$

Further, it is simple to show that

$$3m[\overline{u^4} + (m-1)\overline{u^2 v^2} - m\sigma^4] = 3\left[\overline{\left(\sum_{i=1}^{m} u_i^2 \right)^2} - \left(\overline{\sum_{i=1}^{m} u_i^2} \right)^2 \right],$$
$$(28)$$

which is manifestly non-negative.

This result also gives the condition necessary in order to have both $\mathcal{O}_2(k)$ and $\mathcal{O}_4(k)$ identically vanishing: once Eq. (26) for the conservation of the center of mass of each cloud of the system is assumed, in order to make the variance in Eq. (28) vanish one requires also that

$$\sum_{i=1}^{m} u_i^2 = m\sigma^2 \qquad (29)$$

for each cloud, with probability 1.

In summary, in order to obtain with this algorithm a particle distribution with a SF of order larger than k^4 at small k, one has to satisfy exactly the following two conditions: (i) every cloud must have the same displacement of its center of mass from the initial point of the cloud or mother position [and, in particular, if $p(u)=p(-u)$, the center of mass of the cloud must coincide with the position of the mother particle] and (ii) every cloud must have the same inertial moment (or second moment of its mass dispersion) with respect to the initial point. The value of this inertial moment is fixed by the second moment of $p(u)$ and m as $m\sigma^2$.

This analysis can be continued in order to determine the conditions needed to obtain a $S_\rho(k)$ of order higher than k^{2n} at small k, for any integer n. The result is simply that in order to obtain this goal one has to fix the first n moments of the mass dispersion

$$\sum_{i=1}^{m} u_i^j = c_j \quad \text{with } j = 1, \ldots, n, \qquad (30)$$

where the constants c_j are determined by the jth moment of $p(u)$ and m as $c_j = m\overline{u^j}$. Clearly this gives n conditions, and consequently one has to have at least $m=n$ particles in each cloud in order to make the requirements given by Eq. (30) realizable. For $m=n$ there may be a single nontrivial solution to the constraints—i.e., a unique choice of displacements. In this case the generated distribution will be a lattice with basis, with a SF which again vanishes in a finite region around $k=0$. For $m>n$ the set of constraints may be satisfied (for some range of values of the constants c_j) while leaving free $(m-n)$ degrees of freedom. These may be then fixed stochastically, leading generally to a stochastic particle distribution[7] with a leading nonzero term at $\mathcal{O}_{2(n+1)}(k)$. For $n=2$, for instance, we need at least two particles. To fix the center of mass of the pair at the lattice site, and its second mass mo-

ment to a given value c_2, clearly fixes deterministically the points to lie at $\pm\sqrt{c_2/2}$. Taking three particles one can instead satisfy the constraints fixing one degree of freedom stochastically: placing one point at u with probability $p(u)$, the position u' of a second point is determined by solving the quadratic equation $(u')^2 + uu' + (u^2 - c_2/2) = 0$, and finally a third point is placed at $-(u+u')$. Note that the existence of a solution to the quadratic equation places a strict upper bound on u, $u \leq \sqrt{2c_2/3}$. Thus the probability $p(u)$ necessarily has finite support (and cannot in particular be Gaussian) which is proportional to $\sqrt{c_2}$. It is clear that this is a general requirement for any algorithm of this kind producing a SF with a leading small-k behavior k^n with $n>4$: in order to make the coefficient of the k^4 term vanish the second moment of the mass dispersion of the cloud must be limited with probability 1. Since the displacement of any particle contributes in proportion to its square, the probability of displacement larger than $\sqrt{c_2}$ must be zero.

B. Generalization to $d>1$

In dimensions higher than 1 the problem is essentially the same. The analysis is, however, considerably more complicated because of the vectorial nature of the displacements.

Let us first consider the conditions required to make the terms $\mathcal{O}(k^2)$ and $\mathcal{O}(k^4)$ vanish. Fixing the center of mass in d dimensions gives d scalar equations

$$\sum_{i=1}^{m} u_i^{(\mu)} = 0,$$

where $u_i^{(\mu)}$ is the μth (with $\mu = 1, \ldots, d$) component of the displacement of the ith particle of the cloud. To satisfy this condition nontrivially evidently requires that there be at least two particles in each cloud. Fixing the second moments of the mass dispersion of the cloud gives $d(d+1)/2$ scalar equations—i.e., d equations of the form

$$\sum_{i=1}^{m} [u_i^{(\mu)}]^2 = a_{\mu\mu} > 0 \quad \text{for } \mu = 1, \ldots, d,$$

with $a_{\mu\mu} = m\overline{[u_i^{(\mu)}]^2}$, and $d(d-1)/2$ equations of the form

$$\sum_{i=1}^{m} u_i^{(\mu)} u_i^{(\nu)} = a_{\mu\nu} \quad \text{for } 1 \leq \mu < \nu \leq d,$$

with $a_{\mu\nu} = m\overline{u^{(\mu)} u^{(\nu)}}$. Therefore to obtain a SF $S_\rho(\mathbf{k})$ of order larger than fourth order at small k imposes $[d+d(d+1)/2] = d(d+3)/2$ scalar constraints on the displacements. This counting of constraints may be continued to higher orders, determining the number of conditions, $\mathcal{N}(n,d)$, which must be satisfied to obtain, in d dimensions, an $S_\rho(\mathbf{k})$ vanishing faster than k^{2n} at small k. Noting that all the moments of given order, say, l, of the mass dispersion constitute a fully symmetric l-rank tensor in d dimensions, which has $\binom{d+l-1}{l}$ independent components, we find

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[7]Here we mean by "stochastic" that there is a nonempty compact domain of $\mathbf{k}$ space in which $S_\rho(\mathbf{k})$ is continuous and strictly positive.
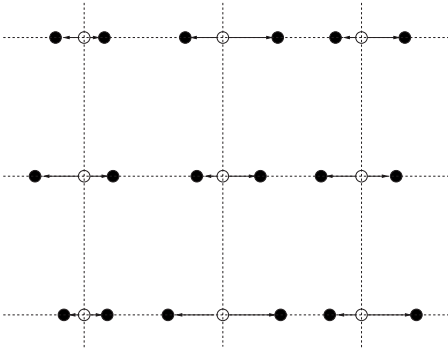
FIG. 2. Pictorial representation of the generation of a cloud process starting from a lattice, in $d=2$, with a "dimensional reduction" to $d=1$ in which the two particles in each cloud are displaced along the same direction.

$$\mathcal{N}(n,d) = \sum_{l=1}^{n} \binom{d+l-1}{l}.$$

Generalizing the reasoning for the case $d=1$, one might then be tempted to conclude that, since each particle brings d degrees of freedom, the minimal number m of particles per cloud required for a given $S_\rho(\mathbf{k})$ of order larger than k^{2n} at small k is $\mathcal{N}(n,d)/d$. This conclusion is, however, not correct in that it tells us the number of particles required to satisfy such conditions for *arbitrary* physical values of the mass moments. To make the coefficients in the expansion of $S_\rho(\mathbf{k})$ vanish up to some order, while remaining nonvanishing at subsequent orders, we require only a number of particles sufficient to allow us to fix a *set* of physical values of the mass moments up to a certain order, while allowing higher moments to vary. Put another way, by imposing additional symmetries or constraints on the PDF of the displacements, we can reduce the number of nontrivial constraints (i.e., equations), reducing the others to simple identities. The following example illustrates this point trivially: in $d>1$ we can always make a cloud lattice by putting together an infinite number of one-dimensional cloud lattices—i.e., by constraining the displacements of the particles to lie along a chosen axis of the lattice. The calculation in $d=1$ then remains valid, as all the additional constraints on the mass moments of the clouds with components in orthogonal directions are trivially satisfied. The same is true in fact if the displacements are in an arbitrary (but fixed) direction. The number m of particles per cloud required to obtain a SF with given leading order then remains the same as in $d=1$. One can also evidently consider less radical "dimensional reductions," taking in d dimensions the displacements of particles in each cloud only in a hyperplane of dimension smaller than d (see Fig. 2).

Even without such a reduction to a lower-dimensional problem, it is easy to give examples in $d>1$ which satisfy the constraints required to make all terms of the SF up to a certain order $2n$ vanish, with much fewer than $\mathcal{N}(n,d)/d$ particles and which do not have the feature of the above examples of breaking the statistical spatial isotropy of the clouds. Consider, for example, the case that each cloud con-
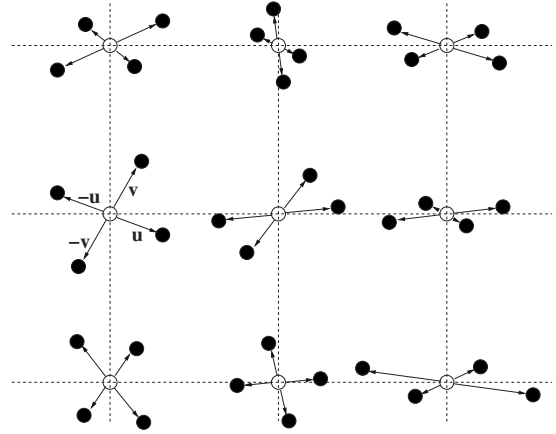

FIG. 3. Pictorial representation of the generation of a lattice cloud process, in $d=2$, in which each cloud is symmetric with respect to its lattice site.

tains an *even* number of particles, arranged symmetrically with respect to the center of mass (see Fig. 3). From the derivation we have given above, it follows immediately that $S_\rho(\mathbf{k}) \propto k^\gamma$ at small k with $\gamma \geq 4$ (as the inversion symmetry gives automatically the conservation of the center of mass). As shown above the coefficients of the term proportional to k^4 will vanish if the second-order moments of the mass distribution are cloud independent. This can be attained, for example, in d dimensions, by taking d pairs of particles arranged symmetrically about, and equidistant from, the origin and all mutually orthogonal. This gives a second moment of the mass dispersion of the cloud which is proportional to the identity matrix and therefore invariant under a random rotation $R \in SO(d)$ of the whole configuration. Further, it is possible to show (see the Appendix for detail) that, because of the imposed inversion symmetry, the terms proportional to k^6 vanish identically. Thus, placing such a cloud with an orientation chosen randomly at each site, one obtains a leading nonzero term of order k^8. This term is nonzero because the (tensor) fourth mass moment is not invariant under rotation of the configuration. Further, if the stochastic process determining the orientation is statistically isotropic, the SF at small k reflects this isotropy and is a function of k only (rather than the vector $\mathbf{k}$). It is simple to check that the number of required particles for this algorithm $2d$ is less, for any d, than the number $\mathcal{N}(3,d)/d$ given above.

V. CORRELATED CLOUDS

We now consider the case in which also displacements applied to particles belonging to different clouds may be arbitrarily correlated. In the notation introduced above for the joint PDFs of the displacements, this means that we now assume, at least for some separation of mother particles $\mathbf{x}$, that $p_d(\mathbf{u},\mathbf{v},\mathbf{x}) \neq p(\mathbf{u})p(\mathbf{v})$. Further, we make the following natural assumptions:

$$p_d(\mathbf{u},\mathbf{v};\mathbf{x}) = p_d(\mathbf{v},\mathbf{u};-\mathbf{x}),$$

$$\lim_{x\to\infty} p_d(\mathbf{u},\mathbf{v};\mathbf{x}) = p(\mathbf{u})p(\mathbf{v}). \tag{31}$$

We have, of course, also the consistency condition

$$\int d^d v \; p_d(\mathbf{u}, \mathbf{v}; \mathbf{x}) = p(\mathbf{u}) \quad \forall \; \mathbf{x}. \quad (32)$$

Note also that, strictly speaking, the displacement field is not continuous as a function of spatial separation because the correlation $\mathbf{u} \cdot \mathbf{v}(\mathbf{x})$ between two displacements applied to two spatial point at vector distance $\mathbf{x}$ in general does not converge to $\overline{u^2}$ for $\mathbf{x} \to \mathbf{0}$. The natural choice for this limit is $p_d(\mathbf{u}, \mathbf{v}; \mathbf{x} \to \mathbf{0}) = p_s(\mathbf{u}, \mathbf{v})$. How precisely this limit is taken will not, however, be of importance for our main application concerning a regular lattice distribution of the centers of the clouds, as in this case the distance between two different centers is always different from zero.

In order to find the expression for $S_\rho(\mathbf{k})$, once $S_n(\mathbf{k})$ and $p_d(\mathbf{u}, \mathbf{v}; \mathbf{x})$ are given, we have to go back to Eq. (6). From this it is simple to show that

$$\frac{\overline{|\tilde{\rho}(\mathbf{k})|^2}}{N} = 1 + (m-1)\tilde{p}_s(\mathbf{k}, -\mathbf{k})$$

$$+ \frac{m^2}{N} \sum_{i \neq j}^{1,M} e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} \tilde{p}_d(\mathbf{k}, -\mathbf{k}; \mathbf{x}_{ij}),$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ and

$$\tilde{p}_d(\mathbf{k}, -\mathbf{k}; \mathbf{x}) = \int d^d u \; d^d v \; p_d(\mathbf{u}, \mathbf{v}; \mathbf{x}) e^{-i\mathbf{k}\cdot(\mathbf{u}-\mathbf{v})}.$$

Note that, analogously to $\tilde{p}_s(\mathbf{k}, -\mathbf{k})$, the function $\tilde{p}_d(\mathbf{k}, -\mathbf{k}; \mathbf{x})$ is the characteristic function of the stochastic vector $\mathbf{w}(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{0})$—i.e., of the difference between the displacements applied to two particles belonging to clouds whose mother particles are separated by $\mathbf{x}$. In other words, if $\phi_d(\mathbf{w}; \mathbf{x})$ is the PDF of $\mathbf{w}(\mathbf{x})$, we have

$$\tilde{p}_d(\mathbf{k}, -\mathbf{k}; \mathbf{x}) = \tilde{\phi}_d(\mathbf{k}; \mathbf{x}),$$

where $\tilde{\phi}_d(\mathbf{k}; \mathbf{x}) = \mathrm{FT}[\phi_d(\mathbf{w}; \mathbf{x})]$. In order to perform the average $\langle \cdots \rangle$, we recall that in general for any function $f(\mathbf{x}, \mathbf{y})$ of two spatial variables one has

$$\left\langle \sum_{i \neq j}^{1,M} f(\mathbf{x}_i, \mathbf{x}_j) \right\rangle = n_0^2 \int \int_V d^d x \; d^d y \; f(\mathbf{x}, \mathbf{y})[1 + h_n(\mathbf{x} - \mathbf{y})],$$

where $h_n(\mathbf{x})$ is the two-point correlation function as defined in Eq. (3). Using this relation, we obtain

$$\left\langle \sum_{i \neq j}^{1,M} e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} \tilde{\phi}_d(\mathbf{k}; \mathbf{x}_{ij}) \right\rangle$$

$$= n_0^2 \int \int_V d^d x \; d^d y \; e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tilde{\phi}_d(\mathbf{k}; \mathbf{x} - \mathbf{y})[1 + h_n(\mathbf{x} - \mathbf{y})].$$

$$(33)$$

Taking the limit $V \to \infty$ with $N/V = \rho_0$ and $m \geq 2$ fixed, we arrive at our result

$$S_\rho(\mathbf{k}) = 1 + (m-1)\tilde{\phi}_s(\mathbf{k}) + \rho_0 \int d^d x \; e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\phi}_d(\mathbf{k}; \mathbf{x})[1 + h_n(\mathbf{x})]$$

$$- (2\pi)^d \rho_0 \delta(\mathbf{k}). \quad (34)$$

Note that the case $m=1$ can be included in Eq. (34) by considering, only for this value of m, $p_d(\mathbf{u}, \mathbf{v}; \mathbf{x})$ to be spatially continuous—i.e., converging to $p(\mathbf{u})\delta(\mathbf{u}-\mathbf{v})$ for $\mathbf{x} \to \mathbf{0}$. The result, Eq. (34), for $m=1$ then agrees exactly with the analogous equation found in [14] for the PS of a particle distribution after the application of a correlated displacement field.

We will not analyze here the small-$\mathbf{k}$ expansion of Eq. (34), but simply note that such an analysis can be done easily by following the steps for the study of the similar equation for the case $m=1$ in [14].

Let us now give the special expression of Eq. (34) for the specific case in which $n(\mathbf{x})$ is a regular lattice point distribution.

A. Correlated cloud lattice

When $n(\mathbf{x})$ is a regular lattice,

$$h_n(\mathbf{x}) = \frac{1}{n_0} \sum_{\mathbf{R} \neq 0} \delta(\mathbf{x} - \mathbf{R}) - 1,$$

where $\mathbf{R}$ runs over all the lattice vectors except $\mathbf{R} = \mathbf{0}$. We therefore have

$$S_\rho(\mathbf{k}) = 1 + (m-1)\tilde{\phi}_s(\mathbf{k}) + m \sum_{\mathbf{R} \neq 0} e^{-i\mathbf{k}\cdot\mathbf{R}} \tilde{\phi}_d(\mathbf{k}; \mathbf{R})$$

$$- (2\pi)^d \rho_0 \delta(\mathbf{k}). \quad (35)$$

This formula is a good starting point for the study of the small-$\mathbf{k}$ behavior of $S_\rho(\mathbf{k})$ for a cloud lattice for different choices of $p_d(\mathbf{u}, \mathbf{v}; \mathbf{R})$ and m. We now use the following chain of identities to rewrite Eq. (34) in a more useful form:

$$(2\pi)^d \rho_0 \delta(\mathbf{k}) = (2\pi)^d \rho_0 \left[\sum_{\mathbf{H}} \delta(\mathbf{k} - \mathbf{H}) - \sum_{\mathbf{H} \neq 0} \delta(\mathbf{k} - \mathbf{H}) \right]$$

$$= m \sum_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} - m S_n(\mathbf{k}), \quad (36)$$

where we have used the definition (22) of $S_n(\mathbf{k})$ for a regular lattice and the lattice identity

$$\sum_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} = (2\pi)^d n_0 \sum_{\mathbf{H}} \delta(\mathbf{k} - \mathbf{H}). \quad (37)$$

By using Eq. (36), we rewrite Eq. (35) as

$$S_\rho(\mathbf{k}) = (m-1)[\tilde{\phi}_s(\mathbf{k}) - 1] + m \sum_{\mathbf{R} \neq 0} e^{-i\mathbf{k}\cdot\mathbf{R}} [\tilde{\phi}_d(\mathbf{k}; \mathbf{R}) - 1]$$

$$+ m S_n(\mathbf{k}). \quad (38)$$

Note that $S_n(\mathbf{k})$ vanishes identically in the first Brillouin zone, so that it does not contribute to the small-$\mathbf{k}$ expansion of Eq. (38). Therefore, assuming all the moments of the displacements to be finite, in this region of k space we can write

$$S_\rho(\mathbf{k}) = (m-1) \sum_{l=1}^\infty (-i)^l \frac{\overline{(\mathbf{k}\cdot\mathbf{w}_s)^l}}{l!}$$

$$+ m \sum_{\mathbf{R}\neq 0} e^{-i\mathbf{k}\cdot\mathbf{R}} \sum_{l=1}^\infty (-i)^l \frac{\overline{[\mathbf{k}\cdot\mathbf{w}_d(\mathbf{R})]^l}}{l!}, \quad (39)$$

where we have denoted, respectively, $\mathbf{w}_s$ the relative displacement of two different particles belonging to the same cloud and $\mathbf{w}_d(\mathbf{R})$ the relative displacement of two particles belonging to two clouds whose centers are separated by the lattice vector $\mathbf{R}$. From this formula we can deduce the conditions on the two-displacement correlations to have a given small-k SF for the resulting particle distribution.

B. Example: Lattice of random correlated dipoles

Let us consider the following example as an application of Eq. (39). Each particle on a perfect lattice is split into two particles, and the following displacements are applied:

$$\mathbf{u}_1(\mathbf{R}) = +\eta(\mathbf{R}), \quad \mathbf{u}_2(\mathbf{R}) = -\eta(\mathbf{R}), \quad (40)$$

where $\eta(\mathbf{R})$ is a random vector at each $\mathbf{R}$ specified by a lattice translationally invariant correlated stochastic process. The average over all the realizations of the displacements of a function $X(\{\eta(\mathbf{R})\})$ of the displacements may be written as the functional integral

$$\overline{X} = \prod_{\mathbf{R}} \int d^d \eta(\mathbf{R}) \mathcal{P}(\{\eta(\mathbf{R})\}) X(\{\eta(\mathbf{R})\}). \quad (41)$$

We assume that the joint probability density function of all the displacements $\mathcal{P}(\{\eta(\mathbf{R})\})$ is invariant under any lattice translation. Moreover it is simple to show, given our symmetry assumption for the displacements, that we can take $\mathcal{P}(\{\eta(\mathbf{R})\})$ to be invariant under the change of sign of any *individual* $\eta(\mathbf{R})$. This ensures that $p_d(\mathbf{u},\mathbf{v};\mathbf{x})$ is well defined as required in our derivation; i.e., the joint probability for displacements to two particles at different sites is the same for all couples and depends only on their relative separation. With this assumption it follows that all odd powers ℓ in the sums in Eq. (39) vanish. Calculating the contribution at second order in $\mathbf{k}$ we have

$$\overline{[\mathbf{k}\cdot\mathbf{w}_s]^2} = 4\overline{[\mathbf{k}\cdot\eta]^2} \quad (42)$$

and

$$\overline{[\mathbf{k}\cdot\mathbf{w}_d(\mathbf{R})]^2} = \int d^3\eta(\mathbf{0}) d^3\eta(\mathbf{R}) p_d[\eta(\mathbf{0}),\eta(\mathbf{R});\mathbf{R}]$$

$$\times \{\mathbf{k}\cdot[\eta(\mathbf{0})-\eta(\mathbf{R})]\}^2 = 2\overline{[\mathbf{k}\cdot\eta]^2}. \quad (43)$$

The latter result uses the fact that the two-point correlation function $\overline{\eta(\mathbf{0})\cdot\eta(\mathbf{R})}=0$, because of the assumed inversion symmetry. Using these results in Eq. (39) together with the identity (37), one finds that the two contributions cancel in the first Brillouin zone to give zero.

At next nontrivial order, fourth order in $\mathbf{k}$, we find

$$\overline{[\mathbf{k}\cdot\mathbf{w}_s]^4} = 16\overline{[\mathbf{k}\cdot\eta]^4} \quad (44)$$

and

$$\overline{[\mathbf{k}\cdot\mathbf{w}_d(\mathbf{R})]^4} = \int d^d\eta(\mathbf{0})d^d\eta(\mathbf{R}) p_d[\eta(\mathbf{0}),\eta(\mathbf{R});\mathbf{R}]$$

$$\times \{\mathbf{k}\cdot[\eta(\mathbf{0})-\eta(\mathbf{R})]\}^4$$

$$= 2\overline{[\mathbf{k}\cdot\eta]^4} + 6\overline{[\mathbf{k}\cdot\eta(\mathbf{0})]^2[\mathbf{k}\cdot\eta(\mathbf{R})]^2}. \quad (45)$$

Using again the identity (37), we obtain the leading nontrivial contribution to the PS which may be written

$$S_\rho(\mathbf{k}) \simeq \frac{1}{2} k_\alpha k_\beta k_\gamma k_\delta \sum_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} \overline{\eta_\alpha(\mathbf{0})\eta_\beta(00)\eta_\gamma(\mathbf{R})\eta_\delta(\mathbf{R})},$$

$$(46)$$

where we adopted the sum convention on the repeated index $\alpha,\beta,\gamma,\delta$. The sum over $\mathbf{R}$ is just the FT (on the lattice) of a two-point correlation function, the behavior of which as $\mathbf{k}\to\mathbf{0}$ depends on the nature of these correlations. The leading small-k behavior of the SF will depend on that of the sum. If the correlations of the dipoles are short range, the sum converges to a positive constant at small k, giving a leading behavior proportional to k^4. If, on the other hand, they are long-range correlated, this sum will diverge as a power-law at small k, with an exponent less than that of the dimension of the space. This will lead to behavior of the overall SF proportional to k^γ with $4-d<\gamma\le 4$. Finally, if the correlations of the dipoles have themselves superhomogeneous properties,[8] one can obtain such a behavior with $\gamma>4$.

VI. SUMMARY AND DISCUSSION

In this paper we have introduced and analyzed a wide class of nontrivial stochastic point processes for which it is possible to write exactly the two-point correlation function and/or SF. They are obtained from a given mother particle distribution, the SF of which is assumed known, by substituting each particle with a cloud of a fixed number m of other particles. The position of the new particles composing the clouds is determined by that of the related mother particle plus a stochastic displacement vector. An important assumption in all our calculations of the SF is that the stochastic process determining the displacements of the particles is independent of the mother distribution; i.e., the displacements of the cloud particles do not depend on the properties, statistical or otherwise, of the mother distribution to which they are applied. In practice, this means that our SF is defined with respect to an ensemble average over two independent ensembles: one describing the realizations of the mother distribution, the other those of the displacement process.

We have distinguished two cases in which (i) only the displacements of different particles belonging to the same cloud may be correlated and (ii) the displacements of particle belonging to different clouds may also be statistically dependent. In both cases we obtain a direct generalization of the relations found in [14] for $m=1$. In the first case, once the average over realizations is taken, the SF of the final particle

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[8]Note that, as the functions $\overline{\eta_\alpha(\mathbf{0})\eta_\beta(\mathbf{0})\eta_\gamma(\mathbf{R})\eta_\delta(\mathbf{R})}$ for $\alpha=\beta$ and $\gamma=\delta$ are non-negative at all $\mathbf{R}$, their lattice Fourier transforms can vanish at $\mathbf{k}=\mathbf{0}$ only if they are identically zero for all $\mathbf{R}$.

distribution is related to the SF of the mother distribution by a local relation in the wave vector $\mathbf{k}$, while the second case leads to a more complex relation. A detailed analysis of case (i) led us to find and to discuss, in the case of an initial lattice mother distribution, the relations linking the exponent of the final SF to the number of conserved mass moments in each cloud: we have seen that such an exponent, if larger or equal to 4, tells us directly which are the locally conserved mass moments in the distribution. When we move to case (ii), the presence of cloud-cloud displacement correlations "interact" with the local mass moments conservation in determining the small-k behavior of the SF of the particle distribution.

One application we have in mind of the results derived here for this class of "stochastically ordered" point processes is in the systematic study of the dynamics of particle systems driven by long-range pair interactions. More specifically, in the case of gravity, it is expected (see, e.g., [7]) that the large-scale fluctuations in an infinite particle system dominate the dynamics of the gravitational clustering for an initial SF with a small-k behavior proportional to k^γ and $\gamma < 4$. This "hierarchical" behavior has been observed numerically for a range of such γ, up to a maximal value of $\gamma = 2$ (see [22] for a recent discussion and further references). No study of the regime of initial conditions $\gamma > 4$ has been performed up to now, as no algorithm has been given in the literature, to our knowledge, which can generate such an initial condition.[9] It is expected that the gravitational clustering will be qualitatively different in this case, with structures being built up from smaller to larger scales. Indeed, the reason why we expect such a difference can be understood easily in the context of our constructions here of such point processes. For $\gamma > 4$ the fluctuations in mass are so suppressed that gravity is effectively "screened" (at least in the initial conditions): in a multipole expansion of the mass far away from a given point in the "uncorrelated cloud lattice" only the leading moment varying from cloud to cloud will contribute (as the contributions from the moments which are fixed will cancel out). If this leading contributing moment is the second moment, this gives an effectively short-range force (decaying as the inverse of the fourth power of the distance).

Our analysis here has been limited to the case of "analytic" exponents for the SF derived from short-tailed PDFs for the displacements. In principle, one can consider also the case in which such PDFs have a long power-law tail. In this case a singular part of the small-k expansion of the related displacement characteristic function arises. This leads (see [8,14]) to a final SF characterized by nonanalytic (e.g., fractional) small-k exponents. We have seen, however, that in the uncorrelated cloud lattice, to attain powers larger than $\gamma = 4$ we must in fact take limited displacements, and thus we necessarily obtain "analytic" exponents (and, in fact, an analytic behavior of the SF at small k). In the case of the correlated cloud lattice, nevertheless, we have given an example (random correlated dipoles) which shows how such nonanalytic powers should be attainable by including appropriate correlations between the clouds.

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[9]Explicit algorithms generating point distribution with $\gamma = 4$ have, on the other hand, been discussed. See, notably, [18,23,24].

We return finally to an important feature of Eq. (34) which we have discussed at some length in Sec. III. This is that the exponent in the small-k scaling behavior of $S_\rho(\mathbf{k})$ cannot be larger than that in $S_n(\mathbf{k})$. As we explained, this can be understood physically as it means that the replacements of particles by clouds cannot make the initial particle distribution more ordered. We underline, however, that this is true given the assumptions we have made and specifically assuming that the stochastic process generating the clouds is independent of the mother distribution. In a forthcoming article with another collaborator [25] we will report results on a related kind of construction of superhomogeneous point processes, starting from tilings of space with equal-volume tiles. In this case it turns out that one can, in certain circumstances, ascribe a cloud of particles to each particle of a given point distribution and as a result increase the exponent γ. The reason why this becomes possible is that the displacements of the particles are not applied independently of the correlation properties of the underlying point process, as we have assumed here. Indeed, to increase the exponent requires that the moments of the clouds be "tuned" appropriately to the tile in which the mother point lies. We note also that, in the case that the initial tiling is taken to be a lattice, the algorithm described coincides with that given here and similar results to those given here are recovered. Details will be reported in [25].

APPENDIX: SYMMETRIC CLOUD LATTICES

In this appendix we study the SF for an uncorrelated cloud process on a lattice for the case that each cloud is symmetric with respect to its own lattice site.

The number density of such a particle distribution can be written as

$$\rho(\mathbf{x}) = \sum_{\mathbf{R}} \sum_{j=1}^{m} \delta(\mathbf{x} - \mathbf{R} - \mathbf{u}_j^{(\mathbf{R})}), \tag{A1}$$

where $\mathbf{R}$ is the lattice site and m is the number of particle per cloud. Since we consider symmetric clouds, m is even and can be written as $m = 2p$ where p is a positive integer. We consider initially a finite lattice with M sites (occupying a corresponding finite volume V) and then send $M \to \infty$ at the end of calculations. Taking the FT of Eq. (A1), we have

$$\widetilde{\rho}(\mathbf{k}, V) = \sum_{\mathbf{R}} \sum_{j=1}^{m} e^{-i\mathbf{k}\cdot(\mathbf{R}+\mathbf{u}_j^{(\mathbf{R})})}. \tag{A2}$$

Now we impose the inversion symmetry for the clouds with respect to their center of mass (i.e., the lattice site). This means that for each particle at $(\mathbf{R}+\mathbf{u})$ there is another particle placed at $(\mathbf{R}-\mathbf{u})$. Therefore for each cloud we can count with j from 1 to $p = m/2$ a set of particles which are not a symmetric image of each other and with $p+j$ their respective symmetric images. Particles in a single cloud with $1 \le j \le p$

can be arbitrarily correlated. Let us call as above $p_s(\mathbf{u}, \mathbf{v})$ the joint PDF of a couple of displacements referred to the set of particles with $j = 1, \ldots, p$ in the same cloud and $p(\mathbf{u})$ the PDF of a single displacement.

Imposing this symmetry of the clouds we can rewrite Eq. (A2) as

$$\widetilde{\rho}(\mathbf{k}, V) = 2 \sum_{\mathbf{R}} e^{-i\mathbf{k}\cdot\mathbf{R}} \sum_{j=1}^{p} \cos(\mathbf{k} \cdot \mathbf{u}_j^{(\mathbf{R})}). \quad \text{(A3)}$$

Then taking the squared modulus we obtain

$$
\begin{aligned}
|\widetilde{\rho}(\mathbf{k}, V)|^2 &= 4 \sum_{\mathbf{R},\mathbf{R}'} e^{-i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} \sum_{l,j}^{1,p} \cos(\mathbf{k} \cdot \mathbf{u}_j^{(\mathbf{R})}) \cos(\mathbf{k} \cdot \mathbf{u}_l^{(\mathbf{R}')}) \\
&= 4 \Bigg\{ \sum_{\mathbf{R}} \sum_{j=1}^{p} [\cos(\mathbf{k} \cdot \mathbf{u}_j^{(\mathbf{R})})]^2 \\
&\quad + \sum_{\mathbf{R}} \sum_{l \neq j}^{1,p} \cos(\mathbf{k} \cdot \mathbf{u}_j^{(\mathbf{R})}) \cos(\mathbf{k} \cdot \mathbf{u}_l^{(\mathbf{R})}) \\
&\quad + \sum_{\mathbf{R} \neq \mathbf{R}'} e^{-i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} \sum_{l,j}^{1,p} \cos(\mathbf{k} \cdot \mathbf{u}_j^{(\mathbf{R})}) \cos(\mathbf{k} \cdot \mathbf{u}_l^{(\mathbf{R}')}) \Bigg\}.
\end{aligned}
\quad \text{(A4)}
$$

We can now take the average over the displacements. In order to do this we recall that we are assuming that the displacements related to particles belonging to different clouds are uncorrelated. Consequently the first and third terms in Eq. (A4) have to be averaged over only $p(\mathbf{u})$, while the second one as to be averaged over $p_s(\mathbf{u}, \mathbf{v})$ containing all the two-displacements correlators in a single cloud. This gives

$$
\begin{aligned}
\langle |\widetilde{\rho}(\mathbf{k}, V)|^2 \rangle &= 4Mp \langle [\cos(\mathbf{k} \cdot \mathbf{u})]^2 \rangle \\
&\quad + 4Mp(p-1) \langle \cos(\mathbf{k} \cdot \mathbf{u}) \cos(\mathbf{k} \cdot \mathbf{v}) \rangle \\
&\quad + 4p^2 \sum_{\mathbf{R} \neq \mathbf{R}'} e^{-i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} \langle \cos(\mathbf{k} \cdot \mathbf{u}) \rangle^2 \\
&= 2N \langle [\cos(\mathbf{k} \cdot \mathbf{u})]^2 \rangle \\
&\quad + N(m-2) \langle \cos(\mathbf{k} \cdot \mathbf{u}) \cos(\mathbf{k} \cdot \mathbf{v}) \rangle \\
&\quad - Nm \langle \cos(\mathbf{k} \cdot \mathbf{u}) \rangle^2 + \text{l.t.}, \quad \text{(A5)}
\end{aligned}
$$

where $N = Mm$ is the total number of particles and "l.t." indicates a "lattice term" which is proportional to the lattice SF (and which therefore does not contribute around $\mathbf{k} = \mathbf{0}$). In performing the last step of Eq. (A5) we have used the simple identity

$$\sum_{\mathbf{R} \neq \mathbf{R}'} e^{-i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} = \sum_{\mathbf{R},\mathbf{R}'} e^{-i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} - M,$$

which gives rise to the third term of the last step. Now we use the definition of the SF:

$$S_\rho(\mathbf{k}) = \lim_{M \to \infty} \frac{\langle |\widetilde{\rho}(\mathbf{k}, V)|^2 \rangle}{N} - (2\pi)^d \rho_0 \delta(\mathbf{k}),$$

which gives

$$
\begin{aligned}
S_\rho(\mathbf{k}) &= 2 \langle [\cos(\mathbf{k} \cdot \mathbf{u})]^2 \rangle + (m-2) \langle \cos(\mathbf{k} \cdot \mathbf{u}) \cos(\mathbf{k} \cdot \mathbf{v}) \rangle \\
&\quad - m \langle \cos(\mathbf{k} \cdot \mathbf{u}) \rangle^2 + \text{l.t.}
\end{aligned}
\quad \text{(A6)}
$$

It is simple to verify that, as expected, $S_\rho(\mathbf{0}) = 0$. We now expand Eq. (A6) in power series of $\mathbf{k}$ and study it order by order:

$$
\begin{aligned}
S_\rho(\mathbf{k}) &= \sum_{a,b}^{0,\infty} \frac{(-1)^{a+b}}{(2a)!(2b)!} [2 \langle (\mathbf{k} \cdot \mathbf{u})^{2(a+b)} \rangle + (m-2) \\
&\quad \times \langle (\mathbf{k} \cdot \mathbf{u})^{2a} (\mathbf{k} \cdot \mathbf{v})^{2b} \rangle - m \langle (\mathbf{k} \cdot \mathbf{u})^{2a} \rangle \langle (\mathbf{k} \cdot \mathbf{u})^{2b} \rangle].
\end{aligned}
\quad \text{(A7)}
$$

We see that only even powers of k are present. Let us call $\mathcal{O}_{2n}(k)$ the term of order k^{2n} in the series above. We see immediately that

$$\mathcal{O}_2(k) = 0$$

as all the clouds conserve the center of mass at their lattice site for symmetry. Therefore the first nontrivial term is $\mathcal{O}_4(k)$, which after some manipulation can be written as

$$\mathcal{O}_4(k) = \frac{1}{2} [\langle (\mathbf{k} \cdot \mathbf{u})^4 \rangle + (p-1) \langle (\mathbf{k} \cdot \mathbf{u})^2 (\mathbf{k} \cdot \mathbf{v})^2 \rangle - p \langle (\mathbf{k} \cdot \mathbf{u})^2 \rangle^2]. \quad \text{(A8)}$$

The next term is

$$
\begin{aligned}
\mathcal{O}_6(k) &= -\frac{1}{12} [\langle (\mathbf{k} \cdot \mathbf{u})^6 \rangle + (p-1) \langle (\mathbf{k} \cdot \mathbf{u})^4 (\mathbf{k} \cdot \mathbf{v})^2 \rangle \\
&\quad - p \langle (\mathbf{k} \cdot \mathbf{u})^4 \rangle \langle (\mathbf{k} \cdot \mathbf{u})^2 \rangle].
\end{aligned}
\quad \text{(A9)}
$$

Both terms are in general nonzero. In order to find the condition for $\mathcal{O}_4(k)$ to vanish, we rewrite it as

$$
\begin{aligned}
\mathcal{O}_4(k) &= \frac{1}{2} \{ \langle [(\mathbf{k} \cdot \mathbf{u})^2 - \langle (\mathbf{k} \cdot \mathbf{u})^2 \rangle]^2 \rangle + (p-1) \langle [(\mathbf{k} \cdot \mathbf{u})^2 \\
&\quad - \langle (\mathbf{k} \cdot \mathbf{u})^2 \rangle][(\mathbf{k} \cdot \mathbf{v})^2 - \langle (\mathbf{k} \cdot \mathbf{v})^2 \rangle] \rangle \}.
\end{aligned}
\quad \text{(A10)}
$$

In any cloud there are $p = m/2$ stochastic displacements which constitute a closed set of p symmetrically correlated variables (i.e., the correlation between any pair of these displacement is constant and there is no correlation with displacements in other clouds). Since the correlation matrix of the random variables $[(\mathbf{k} \cdot \mathbf{u}_j)^2 - \langle (\mathbf{k} \cdot \mathbf{u})^2 \rangle]$, with $j = 1, \ldots, p$ in a single cloud, has to be, as all the correlation matrix of a set of random variables, positive definite (see above), we always have $\mathcal{O}_4(k) \geq 0$. This can be seen in a more intuitive way by noticing what follows:

$$\left\langle \left\{ \sum_{j=1}^{p} [(\mathbf{k} \cdot \mathbf{u}_j)^2 - \langle (\mathbf{k} \cdot \mathbf{u}_j)^2 \rangle] \right\}^2 \right\rangle = 2p\mathcal{O}_4(k), \quad \text{(A11)}$$

which is a variance and consequently manifestly non-negative. Equation (A11) also implies that $\mathcal{O}_4(k)$ vanishes if and only if

$$\left\langle \left\{ \sum_{j=1}^{p} \left[(\mathbf{k} \cdot \mathbf{u}_j)^2 - \langle (\mathbf{k} \cdot \mathbf{u}_j)^2 \rangle \right] \right\}^2 \right\rangle = 0,$$

i.e., if with probability 1 we have

$$\sum_{j=1}^{p} (\mathbf{k} \cdot \mathbf{u}_j)^2 = p \langle (\mathbf{k} \cdot \mathbf{u})^2 \rangle.$$

This is just the "conservation law" of the second moment of the mass dispersion of the clouds in the direction of $\mathbf{k}$. As the orientation of $\mathbf{k}$ is arbitrary, this means that, in order to have $\mathcal{O}_4(k) = 0$ identically, the second moment of the mass of the clouds must be conserved as a tensor $I_{\mu\nu} = \Sigma_{j=1}^{p} u_j^{(\mu)} u_j^{(\nu)}$ with $\mu, \nu = 1, \ldots, d$.

We now analyze $\mathcal{O}_6(k)$ which is given by Eq. (A9). First of all we note that

$$\left\langle \left[\sum_{j=1}^{p} (\mathbf{k} \cdot \mathbf{u}_j)^4 \right] \sum_{l=1}^{p} \left[(\mathbf{k} \cdot \mathbf{u}_l)^2 - \langle (\mathbf{k} \cdot \mathbf{u}_l)^2 \rangle \right] \right\rangle = -12 p \mathcal{O}_6(k).$$

$$(A12)$$

But as seen above, if $\mathcal{O}_4(k) = 0$, the second sum in Eq. (A12) vanishes identically, and therefore we can conclude that when $\mathcal{O}_4(k) = 0$ identically also $\mathcal{O}_6(k) = 0$ automatically and the dominant term in $S_\rho(k)$ becomes $\mathcal{O}_8(k)$.

One can continue this analysis further and show, after some more involved algebra, that the dominant term in the small-k expansion of the SF is of order k^{4n} with n integer and next-order terms are proportional to k^{4n+2q} with q again an integer. The exponent n depends on the order to which the moments of the mass dispersion of the clouds are conserved.

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